Ramsey Graphs Cannot Be Defined by Real Polynomials

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ABSTRACT

Let $P(x,y,n)$ be a real polynomial and let $\{G_n\}$ be a family of graphs, where the set of vertices of G_n is $\{1,2,\ldots,n\}$ and for $1 \le i < j \le n$ $\{i,j\}$ is an edge of G_n iff $P(i,j,n) > 0$. Motivated by a question of Babai, we show that there is a positive constant c depending only on P such that either G_n or its complement $\bar{G}_n$ contains a complete subgraph on at least $c2^{1/2\sqrt{\log n}}$ vertices. Similarly, either G_n or $\bar{G}_n$ contains a complete bipartite subgraph with at least $cn^{1/2}$ vertices in each color class. Similar results are proved for graphs defined by real polynomials in a more general way, showing that such graphs satisfy much stronger Ramsey bounds than do random graphs. This may partially explain the difficulties in finding an explicit construction for good Ramsey graphs.

1. INTRODUCTION

All graphs considered here are finite, undirected, and simple. A graph G is called p-*Ramsey* if neither G nor its complement $\bar{G}$ contains a complete graph K_p on p vertices. Similarly, G is called (p, q)-*bipartite Ramsey* if neither G nor $\bar{G}$ contains a complete bipartite graph $K_{p,q}$ with classes of vertices of sizes p and q. The well-known theorem of Ramsey ([9], see also, e.g., [8]) asserts that no graph on n vertices is $1/2 \log n$-Ramsey. (Here, and throughout the paper, all logarithms are in base 2, unless otherwise specified.) On the other hand, Erdös proved in [4] that there are graphs on n vertices that are $2 \log n$-Ramsey. This proof was one of the first applications of the probabilistic method in *Combinatorics* and it does not supply an explicit construction of such graphs. In fact, the problem, a prize for

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whose solution is offered by Erdös in [5], of constructing explicitly, for some constant $c > 0$, a family of graphs $\{G_n\}$, where G_n has n vertices and is $c \log n$-Ramsey, is still open, although it received a considerable amount of attention. The best known construction, due to Frankl and Wilson ([7]) supplies graphs on n vertices that are $e^{c\sqrt{\log n \log \log n}}$-Ramsey for some $c > 0$. The situation for bipartite Ramsey graphs is even worse; to the best of our knowledge, although the probabilistic method easily implies the existence of graphs on n vertices that are $(c\log n, c\log n)$-bipartite Ramsey for some $c > 0$, there is no known explicit construction of graphs on n vertices that are, say, $(n^{1/10}, n^{1/10})$-bipartite Ramsey. (It is worth noting, though, that the Paley graphs form an explicit family of graphs on n vertices that are, for example, $(1/4 \log n, (1 + o(1))\, n^{3/4})$-bipartite-Ramsey but the construction of a (p, q)-bipartite-Ramsey graph where both p and q are small seems much more difficult).

In an attempt to understand the difficulty in finding an explicit construction of good Ramsey graphs, i.e., graphs on n vertices that are $c\log n$-Ramsey, Babai conjectured that such graphs cannot be defined by real polynomials. In fact, he made the following conjecture that asserts that such graphs satisfy much stronger Ramsey bounds than do random graphs:

Conjecture 1.1 (Babai [3]). Let $\{G_n\}$ be a family of graphs defined by the real polynomial $P(x,y,n)$ as follows: The set of vertices of G_n is $\{1,2,\ldots,n\}$. For $1 \le i < j \le n$, $\{i,j\}$ is an edge of G_n iff $P(i,j,n) > 0$. Then there is a positive constant c, depending only on P, such that for every n, G_n is not a cn^{ε}-Ramsey graph, where ε is a positive absolute constant.

Notice that if for example, we define $P(x,y,n) = (x - y)^2 - n$, then the resulting graphs G_n will be $O(\sqrt{n})$-Ramsey and hence ε must be at most 1/2 in the above conjecture.

At the moment we are unable to settle this conjecture, as stated. On the other hand, we can prove the underlying idea suggested by it by showing that graphs defined by a polynomial as above cannot be good Ramsey graphs; in fact, G_n is not a $(c\sqrt{n}, c\sqrt{n})$-bipartite-Ramsey graph and it is also not a $c'2^{1/2\sqrt{\log n}}$-Ramsey graph for appropriately chosen positive constants c and c' depending on P. Moreover, a similar result holds even if we allow the vertices to be represented by arbitrary l-tuples of real numbers and allow several real polynomials instead of one. In order to state our precise results we need one more definition.

Definition 1.2. Let $\mathbf{P} = (P_1, P_2, \ldots, P_k)$ be a sequence of k real polynomials, where each $P_m = P_m(x,y,n)$ with $x, y \in \mathbf{R}^l$ is a polynomial in the $2l + 1$ variables $x = (x_1, \ldots, x_l), y = (y_1, \ldots, y_l)$ and n. A graph G_n on n vertices $v_1, \ldots, v_n$ is an 1-P-*polynomial graph* (or simply a $\mathbf{P}$-graph) if there is a sequence $z_1, z_2, \ldots, z_n \in \mathbf{R}^l$ of l-tuples of real numbers and a set S of vectors in $\{-1,0,1\}^k$, such that for each $1 \le i < j \le n$, $\{v_i, v_j\}$ is an edge of G_n iff there is a vector $s = (s_1, \ldots, s_k) \in S$ such that sign $(P_m(z_i, z_j, n)) = s_m$ for all $1 \leq i \leq k$.

Thus, a graph is a polynomial graph if the existence or the nonexistence of each of its edges is determined by the signs of a given set of real polynomials evaluated at the values corresponding to the end points of that edge. In particular, the graphs G_n considered in Conjecture 1.1 are 1-**P** graphs for $\mathbf{P} = (P(x, y, n))$, where the sequence $z_1, \ldots, z_n$ is simply $1, 2, \ldots, n$ and the set S consists of the single one-dimensional vector 1.

In this paper we prove the following three theorems. The first two show that polynomial graphs cannot be good Ramsey graphs, and the third shows that in certain special cases Conjecture 1.1 is valid.

Theorem 1.3. Let $\mathbf{P}$ be a sequence of k real polynomials in x, y and n, where $x, y \in \mathbf{R}^l$. Then there exists a positive constant $c = c(\mathbf{P})$ such that for every $l-\mathbf{P}$-graph G_n on n vertices, G_n is not a $(p, [(n-2p)/(2cp)^l])$-bipartite-Ramsey graph for any $p \geq 1$. In particular, G_n is not a $(c'n^{1/(l+1)}, c'n^{1/(l+1)})$-bipartite-Ramsey graph, for an appropriately chosen constant $c' > 0$.

Theorem 1.4. Let $\mathbf{P}$ be a sequence of k real polynomials in x, y, and n, where $x, y \in \mathbf{R}^l$. Then there exists a positive constant $c = c(\mathbf{P})$ such that every $l-\mathbf{P}$-graph G_n on n vertices is not a $c\, e^{c_l \sqrt{\log n}}$-Ramsey graph, where $c_l > 0$ is a constant depending only on l.

Theorem 1.5. Let $\mathbf{P}$ be a real polynomial in the difference $x - y$ and in n, and let G_n be the graph on the n vertices $\{1, 2, \ldots, n\}$ in which for $1 \leq i < j \leq n$ $\{i, j\}$ is an edge iff $P(i, j, n) > 0$. Then there is a constant $c = c(\mathbf{P}) > 0$ such that for all n G_n is not a $c\sqrt{n}$-Ramsey-graph.

The estimate given in Theorem 1.5 is best possible, up to the value of c, as shown by the polynomial $(x - y)^2 - n$.

Our paper is organized as follows. In Section 2 we show how a theorem of Warren from real algebraic geometry supplies a bound on the number of subgraphs of a certain type of a polynomial graph. In Section 3 we show that this bound easily implies Theorem 1.3. Together with a method similar to the one used by Erdös and Hajnal in [6] it also provides a proof of Theorem 1.4. The proof of Theorem 1.5 is given in Section 1.4. The final Section 5 contains some concluding remarks and open problems.

2. SIGN PATTERNS AND TRACES

Let $\mathbf{Q} = (Q_1, \ldots, Q_m)$ be a sequence of m polynomials in t real variables. For each point $x \in \mathbf{R}^t$, the vector (sign $Q_1(x), \ldots,$ sign $Q_m(x)) \in \{-1, 0, 1\}^m$

is a *sign-vector* of $\mathbf{Q}$. Let $s(Q_1, \ldots, Q_m)$ denote the total number of sign-vectors of $\mathbf{Q}$, as x ranges over all points of $\mathbf{R}^t$. Warren [10] proved an upper bound for the number of sign patterns of $\mathbf{Q}$ that lie in $\{-1,1\}^m$ in terms of t, m, and the degrees of the polynomials $\mathbf{Q}$. His bound can be easily extended to bound $s(Q_1, \ldots, Q_m)$ and give the following estimate:

Lemma 2.1 (Warren [10]; see also [1], [2]). Let $\mathbf{Q} = (Q_1, \ldots, Q_m)$ be a sequence of m polynomials in t real variables. If the degree of each Q_i is at most d and $m \geq t$, then

$$s(Q_1, \ldots, Q_m) \leq \left(\frac{8edm}{t}\right)^t. \quad \blacksquare$$

The next simple corollary of the above lemma is not required for the proofs of our main theorems, but may be interesting in its own right.

Corollary 2.2. Let $\mathbf{P} = (P_1, \ldots, P_k)$ be a sequence of k real polynomials in x, y, and n where $x, y \in \mathbf{R}^l$. Then, there exists a constant $c = c(\mathbf{P}) > 0$ such that the total number of nonisomorphic $l - \mathbf{P}$-graphs on n vertices is at most

$$(cn)^{nl}.$$

Proof. Let $v_1, \ldots, v_n$ be the vertices of an $l - \mathbf{P}$-graph G on n vertices, and let $z_1, \ldots, z_n \in \mathbf{R}^l$ be the values associated with the vertices. Then G is determined by the signs of the $k \cdot \binom{n}{2}$ polynomials $P_s(z_i, z_j, n)$ $(1 \leq s \leq k, 1 \leq i < j \leq n)$ in the $l \cdot n$ real variables that are the coordinates of the vectors z_i. Let d be the maximum degree of a P_s and define $c = (4edk)/l$. The result (for $k\binom{n}{2} \geq ln$) now follows from Lemma 2.1. By increasing c, if necessary we can make sure that the bound will hold for the values of n for which $k\binom{n}{2} < ln$, as well. $\blacksquare$

Observe that any induced subgraph of a $\mathbf{P}$-graph is also a polynomial graph (with slightly different polynomials obtained from those of $\mathbf{P}$ by shifting n). Thus, the last Corollary bounds the number of induced subgraphs with a given number of vertices of any $\mathbf{P}$-graph.

We next prove a similar lemma that will be useful later. If $G = (V, E)$ is a graph, $U \subseteq V$ and $v \in V \backslash U$, then *the trace of v on U* is $\mathrm{tr}(v, U) = \{u \in U : vu \in E\}$, i.e., it is the set of all neighbors of v in U.

Lemma 2.3. Let $\mathbf{P} = (P_1, \ldots, P_k)$ be a sequence of real polynomials in x, y, and n where $x, y \in \mathbf{R}^l$. Then there exists a constant $c = c(\mathbf{P}) > 0$ such that for any $l - \mathbf{P}$-graph $G_n = (V, E)$ and for any subset $U \subseteq V$, the number of distinct traces of vertices in $V\backslash U$ on U is at most $(c|U|)^{l+1}$. Moreover, if U is the set of $|U|$ first vertices of G_n (in the order used in the representation of G_n as a **P**-graph) then the number of distinct traces of vertices in $V\backslash U$ on U is at most $(c|U|)^l$.

Proof. Let $\{z_u : u \in U\}$ be the values in $\mathbf{R}^l$ corresponding to the members of U in the realization of G_n as a **P**-graph. For $v \in V\backslash U$, let $z_v \in \mathbf{R}^l$ be the value corresponding to v. Recall that in the definition of a **P**-graph there is a linear order on the vertices. The trace of v on U is determined by the rank of v in this order as induced on $U \cup \{v\}$ (for which there are $|U| + 1$ possibilities) and by the signs of the $2k \cdot |U|$ polynomials $P_s(z_u, z_v, n)$ and $P_s(z_v, z_u, n)$ $(1 \leq s \leq k, u \in U)$. In these polynomials, n and $\{z_u; u \in U\}$ are fixed, and only z_v varies as v varies. Thus there are only l variables, and the desired bound (for $2k|U| \geq l\}$ follows from Lemma 2.1. By increasing c, if necessary, we can guarantee that the assertion of the lemma will hold for all values of $|U|$. In case U is the set of $|U|$ first vertices of G_n the factor of $|U| + 1$ arising from the rank of v in $U \cup \{v\}$ can be saved. (Also, in this case we may consider only the $k\ |U|$ polynomials $P_s(z_u, z_v, n)$, but this only affects the constant c). ∎

It is worth noting that in the case $l = 1$ (considered in Conjecture 1.1), Warren's Theorem (Lemma 2.1) is not necessary, as in this case we only need a bound for the number of sign patterns of a family of one-variable real polynomials, which follows trivially from the fact that each of them does not have too many distinct real roots.

3. RAMSEY GRAPHS AND BIPARTITE-RAMSEY-GRAPHS

We start this section with the simple derivation of Theorem 1.3 from Lemma 2.3. Let **P** be a sequence of k real polynomials in x, y, and n, where $x, y \in \mathbf{R}^l$. Let $c = c(\mathbf{P}) > 0$ be the constant supplied by Lemma 2.3. Suppose, now that $G = (V, E)$ is an arbitrary l-**P**-graph on n vertices. We have to show that for every $p \geq 1$ G is not $(p, \lceil (n - 2p)/(2cp)^l \rceil)$-bipartite-Ramsey; i.e., either G or its complement $\bar{G}$ contains a complete bipartite graph with classes of vertices of sizes p and $\lceil (n - 2p)/(2cp)^l \rceil$. If $n \leq 2p$ there is nothing to prove. Suppose, thus, that $n > 2p$ and let U be the set of first $2p$ vertices of G_n. By Lemma 2.3, the number of distinct traces of vertices in $V\backslash U$ on U is at most $(2cp)^l$. Hence, there is a set $W \subseteq V$, $|W| \geq (n - 2p)/(2cp)^l$ such that each $w \in W$ has exactly the same trace on U. Put $U_l = \{u \in U; wu \in E$ for all $w \in W\}$ and $U_2 = \{u \in U : wu \notin E$ for all $w \in W\}$. Clearly U is the disjoint union of U_1 and U_2 and hence either $|U_1| \geq p$ or $|U_2| \geq p$ (or both). In the first case, G contains a complete bipartite graph on the classes of vertices U_1 and W. In the second case the complement graph $\bar{G}$ contains a complete bipartite graph on U_2 and W. This completes the proof of Theorem 1.3. ∎

The nonbipartite case, considered in Theorem 1.4, is somewhat more complicated. We first define, by induction, a family $\mathcal{F}$ of perfect graphs as follows. The trivial graph K_1 belongs to $\mathcal{F}$. If $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are two members of $\mathcal{F}$ then their disjoint union, as well as their join (i.e., the graph obtained from their disjoint union by adding all edges $\{v_1v_2 : v_2 \in V_1, v_2 \in V_2\}$) are members of $\mathcal{F}$. Clearly every induced subgraph of a member of $\mathcal{F}$ belongs to $\mathcal{F}$ and one can easily show by induction that all graphs in $\mathcal{F}$ are perfect. Therefore, if a graph $F \in \mathcal{F}$ has m vertices then either F or $\bar{F}$ contains a complete graph on at least $\sqrt{m}$ vertices. (This is because if F contains no $K_{\lceil\sqrt{m}\rceil}$ then it is $\lceil\sqrt{m}\rceil - 1$-colorable and hence contains an independent set of size at least $\sqrt{m}$). We conclude that in order to show that a graph is not a good Ramsey graph is suffices to show that it contains a relatively large induced subgraph from $\mathcal{F}$.

Proposition 3.1. Let $G = (V, E)$ be a graph on n vertices in which for every $U \subseteq V$ the number of distinct traces of vertices in $V \backslash U$ on U is at most $(c|U|)^{l+1}$ where $l \geq 1$ is an integer. Then G contains an induced subgraph from $\mathcal{F}$ on at least

$$\frac{1}{2c} \cdot 2^{\sqrt{2\log n/(l+1)}}$$

vertices.

Proof. We apply induction on n. Since obviously $c \geq 1$ (consider sets U of cardinality 1), and since every graph on at most 3 vertices is in $\mathcal{F}$ there is nothing to prove for $n \leq 16$. Suppose the proposition holds for all $n' < n$ and let us prove it for n $(n > 16)$. Let $G = (V, E)$ be a graph on n vertices satisfying the assumptions. Let f be the maximum cardinality of an induced subgraph of G that belongs to $\mathcal{F}$, and denote by U $(U \subseteq V, |U| = f)$ its set of vertices. Clearly $f \geq 3$. If $f \geq (1/2c)\,2^{\sqrt{2\log n/(l+1)}}$ there is nothing to prove and hence we may assume that

$$f < (1/2c)\,2^{\sqrt{2\log n/(l+1)}}.$$

This and the fact that $c \geq 1$ and $n > 16$ imply that $f < n/2$ and that $1/(cf)^{l+1} > 2^{l+1-\sqrt{2(l+l)\log n}}$ and hence that

$$\left\lceil \frac{n-f}{(cf)^{l+1}} \right\rceil > n \cdot 2^{l-\sqrt{2(l+1)\log n}}. \tag{1}$$

By the assumptions, there are at most $(cf)^{l+1}$ distinct traces on U. Thus, there is a set $W \subseteq V \backslash U$ of cardinality $|W| \geq \lceil (n-f)/(cf)^{l+1} \rceil$ such that each $w \in W$ has the same trace on U. By the induction hypothesis there is a subset Y of W satisfying

$$|Y| \geq \frac{1}{2c} 2^{\sqrt{2\log|W|/(l+1)}} \tag{2}$$

such that the induced subgraph of G on Y belongs to $\mathcal{F}$. Define $U_1 = \{u \in U : uy \in E \text{ for all } y \in Y\}$ and $U_2 = \{u \in U : uy \notin E \text{ for all } y \in Y\}$. Since all vertices of Y have the same trace on U, U is the disjoint union of U_1 and U_2. Moreover, the induced subgraphs of G on $U_1 \cup Y$ and on $U_2 \cup Y$ both belong to $\mathcal{F}$, and at least one of them has at least $f/2 + |Y|$ vertices. By the maximality of f this implies that $f/2 + |Y| \leq f$, i.e., $f \geq 2|Y|$. Combining this with (2), the fact that $|W| \geq \lceil (n-f)/(cf)^{l+1} \rceil$, (1), and the monotonicity of the exponential and the logarithmic function we conclude that

$$f \geq 2|Y| \geq \frac{1}{c} 2^{\sqrt{(2/(l+1))\log(n \cdot 2^{l-\sqrt{2(l+1)\log n}})}} \tag{3}$$

To complete the proof it thus suffices to check that the right hand side of (3) is at least $(1/2c)\, 2^{\sqrt{2\log n/(l+1)}}$, or equivalently, that

$$1 + \sqrt{\frac{2}{l+1}\log(n \cdot 2^{l-\sqrt{2(l+1)\log n}})} \geq \sqrt{\frac{2\log n}{l+1}}.$$

However, this last inequality is valid, since

$$\begin{aligned}
&\sqrt{\frac{2}{l+1}}\left(\sqrt{\log n} - \sqrt{\log(n \cdot 2^{l-\sqrt{2(l+1)\log n}})}\right) \\
&= \sqrt{\frac{2}{l+1}}\left(\sqrt{\log n} - \sqrt{\log n - \sqrt{2(l+1)\log n} + l}\right) \\
&\leq \sqrt{\frac{2}{l+1}}\left(\sqrt{\log n} - \sqrt{\log n - \sqrt{2(l+1)\log n} + \frac{l+1}{2}}\right) \\
&= \sqrt{\frac{2}{l+1}}\left(\sqrt{\log n} - \left(\sqrt{\log n} - \sqrt{\frac{l+1}{2}}\right)\right) = 1,
\end{aligned}$$

as needed. This completes the proof of the proposition. ∎

Corollary 3.2. Let $\mathbf{P}$ be a sequence of k real polynomials in x, y, and n, where $x, y \in \mathbf{R}^l$. Then there exists a positive constant $a = a(\mathbf{P})$ such that every $l - \mathbf{P}$-graph G_n on n vertices is not an $a \cdot 2^{\sqrt{\log n/(2(l+1))}}$-Ramsey graph.

Proof. By Lemma 2.3 there is a positive constant $c = c(\mathbf{P})$ such that the number of distinct traces on any subset U of vertices of G_n is at most $(c|U|)^{l+1}$. By Proposition 3.1, G_n contains an induced perfect graph on $m \geq (1/2c)\, 2^{\sqrt{2\log n/(l+1)}}$ vertices. Hence, either G or $\bar{G}$ contain a complete graph on at least $\sqrt{m}$ vertices, completing the proof. ∎

Corollary 3.2 implies, of course, Theorem 1.4. For the case $l = 1$ it shows that for every $1 - \mathbf{P}$-graph G_n on n vertices either G or $\bar{G}$ contains a complete subgraph on $a\ 2^{(1/2)\sqrt{\log n}}$ vertices, where a is a positive constant depending only on $\mathbf{P}$.

4. POLYNOMIALS IN $(x - y)$

In this section we prove Theorem 1.5 that deals with polynomial graphs defined by a polynomial in the difference $x - y$ and in n. We need the following lemma.

Lemma 4.1. For every integer $k \geq 1$, there is a positive constant $c_k > 0$ such that for any family of k disjoint open intervals $\{(a_i, b_i)\}_{i=1}^k$, where $1 \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_k < b_k$ are $2k$ integers, there is a set $S \subseteq \{1, \dots, b_k\}$ of at least $c_k \prod_{i=1}^k (b_i/a_i)$ integers, such that for any $s, t \in S, s < t$, the differences $t - s$ belongs to $\bigcup_{i=1}^k (a_i, b_i)$.

Proof. We apply induction on k and prove the lemma with $c_k = 1/4^k$. (This estimate can be easily improved; we make no attempt to optimize the constant c_k). For $k = 1$, if $b_1 \leq 4a_1, S$ can be an arbitrary 1-element subset of $\{1, \dots, b_1\}$ (e.g., $S = \{1\}$). If $b_1 > 4a_1$, let S be the set of all positive multiples of $2a_1$, which are smaller than b_1. Then $|S| \geq (b_1/2a_1) - 1 > (1/4)(b_1/a_1)$ and for any $s, t \in S, s < t, a_1 < 2a_1 \leq t - s < t < b_1$, as needed. Suppose the assertion of the lemma holds for k (with $c_k = 1/4^k$), and let us prove it for $k + 1$. Let $\{(a_i, b_i)\}_{i=1}^{k+1}$ be a set of $k + 1$ open intervals, where $1 \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_{k+1} < b_{k+1}$ are integers. By the induction hypothesis there is a set $\bar{S} \subseteq \{1, \dots, b_k\}$ such that $|\bar{S}| \geq (1/4^k) \prod_{i=1}^k (b_i/a_i)$ and such that for every $s, t \in \bar{S}, s < t, t - s \in \bigcup_{i=1}^k (a_i, b_i)$. Consider two possible cases.

Case 1. $b_{k+1} \leq 4a_{k+1}$.

In this case, $|\bar{S}| \geq (1/4^{k+1}) \prod_{i=1}^{k+1} (b_i/a_i) = c_{k+l} \prod_{i=1}^{k+1} (b_i/a_i)$ and hence the set $\bar{S}$ itself can serve as our S.

Case 2. $b_{k+1} > 4a_{k+1}$.

Let T be the set of all nonnegative multiples of $2a_{k+1}$(including 0) which are strictly smaller than $b_{k+1} - a_{k+1}$. Then $|T| \geq (b_{k+1} - a_{k+1})/(2a_{k+1}) > (1/4)(b_{k+1}/a_{k+1})$. Define $S = \bar{S} + T = \{\bar{s} + t : \bar{s} \in \bar{S}, t \in T\}$. Since the difference between any two distinct numbers in T is bigger than the maximum number in $\bar{S}, |S| = |\bar{S}| \cdot |T| \geq (1/4^{k+1}) \prod_{i=1}^{k+1} (b_{k+1}/a_{k+1})$. Also, if $\bar{s}_1 + t_1$ and $\bar{s}_2 + t_2$ are two numbers in $S = \bar{S} + T$ with $\bar{s}_i \in \bar{S}, t_i \in T$, and

$\bar{s}_1 + t_1 < \bar{s}_2 + t_2$, then $(\bar{s}_2 + t_2) - (\bar{s}_1 + t_1) = (\bar{s}_2 - \bar{s}_1) + (t_2 - t_1)$. If $t_2 > t_1$ then

$$(\bar{s}_2 - \bar{s}_1) + (t_2 - t_1) > -b_k + 2a_{k+1} \geq a_{k+1}$$

and

$$(\bar{s}_2 - \bar{s}_1) + (t_2 - t_1) < b_k + b_{k+1} - a_{k+1} \leq b_{k+1}$$

i.e., the difference is in the interval (a_{k+1}, b_{k+1}). Otherwise $t_2 = t_1$ (since if $t_2 < t_1$ then $t_2 + \bar{s}_2 < t_1 + \bar{s}_1$, contradicting the assumption), and then $\bar{s}_2 > \bar{s}_1$ and

$$(\bar{s}_2 - \bar{s}_1) + (t_2 - t_1) = \bar{s}_2 - \bar{s}_1 \in \cup_{i=1}^{k}(a_i, b_i),$$

by the choice of $\bar{S}$.

Thus S satisfies the conclusion of the lemma for $k + 1$ intervals. This completes the induction and the proof. ∎

Proof of Theorem 1.5. Let $P = P(x, y, l)$ be a real polynomial in the difference $x - y$ and in l, and let $\{G_n\}$ be the sequence of graphs where G_n is a graph on the set of n vertices $\{1, 2, \ldots, n\}$ in which for $1 \leq i < j \leq n$ $\{i, j\}$ is an edge iff $P(i, j, n) > 0$. We must show that for every n either G_n or $\bar{G}_n$ contains a complete subgraph on $m \geq c\sqrt{n}$ vertices, where $c = c(\mathbf{P}) > 0$ is a constant depending only on P. Let k be the $(x - y)$-degree of P (i.e., the largest power of $x - y$ appearing in the standard representation of P as a sum of monomials in $x - y$ and l). We claim that for every n either G_n or $\bar{G}_n$ contains a complete subgraph on at least $(1/2^{k+1})\sqrt{n}$ vertices. Indeed, let $g(y - x) = P(x, y, n)$ be the polynomial (in the one variable $y - x$) obtained from $P(x, y, l)$ by substituting $l = n$. If $g(z)$ is identically zero or if it has no real roots in the closed interval $[1, n]$, then G_n is either complete or empty and the claim is trivial. Otherwise, $g(z)$ has at most k real roots with distinct integer parts $v_1 < v_2 < \ldots < v_s (s \leq k)$ in $[1, n]$. These s roots can be used to define $s + 1$ intervals $I_1, \ldots, I_{s+1}$ as follows: $I_1 = [1, \lfloor v_1 \rfloor)$, $I_2 = (\lfloor v_1 \rfloor, \lfloor v_2 \rfloor), I_3 = (\lfloor v_2 \rfloor, \lfloor v_3 \rfloor), \ldots, I_s = (\lfloor v_{s-1} \rfloor, \lfloor v_s \rfloor)$, $I_{s+1} = (\lfloor v_s \rfloor, n)$. Set $J_1 = \{j : 1 \leq j \leq s + 1$ and $P(z)$ is positive for all the integers $z \in I_j\}$, $J_2 = \{j : 1 \leq j \leq s + 1, I_j$ contains at least one integer and $P(z)$ is negative for all the integers $z \in I_j\}$. One can easily check that $\{1, 2, \ldots, s + 1\} = J_1 \cup J_2$ and $J_1 \cap J_2 = \varnothing$. For convenience define $v_0 = 1$, $v_{s+1} = n$ and $r_j = \lfloor v_i \rfloor$ for each j.

By Lemma 4.1 there exists a set $S_1 \subseteq \{1, 2, \ldots, n\}$, $|S_1| \geq (1/4^{|J_1|}) \Pi_{j \in J_1}(r_j/r_{j-1})$ such that for every $s, t \in S_1$, with $s < t$, the difference $t - s$ lies in $\cup_{j \in S_1} I_j$. Therefore $P(s, t, n) > 0$ for all $s, t \in S$, with $s < t$, i.e., S_1 is the set of vertices of a complete graph of G_n. Similarly, Lemma 4.1 implies the existence of a set $S_2 \subseteq \{1, 2, \ldots, n\}$, $|S_2| \geq (1/4^{|J_2|}) \prod_{j\in J_2}(r_j/r_{j-1})$ such that for every $s, t \in S_2$ with $s < t$; $t - s \in \bigcup_{j\in S_2} I_j$ and hence $P(s,t,n) < 0$. This means that S_2 is the set of vertices of an independent set in G_n. Now

$$|S_1| \cdot |S_2| \geq \frac{1}{4^{|J_1|+|J_2|}} \prod_{j\in J_1\cup J_2} \frac{r_j}{r_{j-1}} = \frac{1}{4^{s+1}} \cdot n\,.$$

and thus either S_1 or S_2 has cardinality at least $\sqrt{(1/4^{s+1})n} \geq (1/2^{k+1})\sqrt{n}$. This completes the proof of the theorem. ∎

5. CONCLUDING REMARKS AND OPEN PROBLEMS

1. The assertions of all the results in this paper remain true even if we do not assume that the functions $P_i(x, y, n)$ are polynomials in all variables; the dependence on n may be a nonpolynomial one, as long as there is a bound d such that for every substitution for n the resulting polynomial is of degree at most d in x and y. In fact, we may define a sequence of graphs $\{G_n\}$, where each G_n is defined by a different set of polynomials in x and y, such that all polynomials are of degree at most d and $x, y \in \mathbf{R}^l$. The proofs in the previous sections clearly work for this more general case too.
2. It seems plausible that Conjecture 1.1 holds even with $\varepsilon = 1/2$ (which would then be best possible). In fact, we believe that for every sequence $\mathbf{P} = (P_1, \ldots, P_k)$ of polynomials in the three variables x, y, and n there is a constant $c = c(\mathbf{P}) > 0$ such that no $1 - \mathbf{P}$-graph G_n on n vertices is a $c\sqrt{n}$-Ramsey graph. (Note that by Theorem 1.3 no such graph is $(c'\sqrt{n}, c'\sqrt{n})$-bipartite-Ramsey graph for some $c' = c'(\mathbf{P}) > 0$).

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